$\boxed{m}$ = dim m framing "flavour"

$\bigcirc\!\!\!\!\!n$ = dim n "gauge"

$$\begin{tikzcd}[column sep=small]
\boxed{m_1} \arrow[d, dash] & \cdots & \boxed{m_k} \arrow[d, dash] \\
n_1 & \cdots & n_k
\end{tikzcd}
\rightsquigarrow T^*\mathrm{Rep}(\bigcirc, \square) /\!\!/\!\!/_{\lambda,\theta} GL_{\bigcirc}.$$

more arrows

<u>Ex</u>

framed A_1.

$$\begin{tikzcd}
\boxed{n} \arrow[d, dash] \\
1
\end{tikzcd}
\qquad T^*\mathbb{C}^n /\!\!/\!\!/_{\theta} GL_1 = \mathbb{C}^\times \cong T^*\mathbb{P}^{n-1}, \ \theta \neq 0.$$

$$\Downarrow$$

$$\min(\mathfrak{sl}_n), \ \theta = 0.$$

$$\mathbb{C}^n \times (\mathbb{C}^n)^* \supseteq \mu^{-1}(0) = \{ (v, \varphi) \mid \varphi(v) = 0 \}$$

$$\longmapsto \underbrace{v \otimes \varphi}_{=A} \in \mathbb{C}^n \otimes (\mathbb{C}^n)^* = \mathrm{Mat}_n \mathbb{C}$$

Result: $\mathrm{rk} = 1$, $A^2 = 0$.

$G = SL_n$, B Borel

$$\begin{tikzcd} \boxed{n} \arrow[d, no head] & & & \\ (n-1) \arrow[r, no head] & (n-2) \arrow[r, no head] & \cdots \arrow[r, no head] & (1) \end{tikzcd}$$

$\Rightarrow G/B \cong$ ordinary $\mathcal{M}^{\theta}_{\alpha} Q$

θ suitable

$$\begin{tikzcd} T^* G/B \arrow[d] \cong \mathfrak{M}_{0,\theta}(Q, \alpha) \\ \mathrm{Nil}(\mathfrak{g}) \cong \mathfrak{M}_{0,0}(Q, \alpha). \end{tikzcd}$$

⚠ In general, $T^*G/B \twoheadrightarrow \mathcal{N}il\, \mathfrak{g}$

NOT quiver varieties, $\mathfrak{g}$ not type A ($\mathfrak{sl}_n$)

However, nilp. orbit closures in $\mathfrak{so}_n$, $\mathfrak{sp}_{2n}$ ARE Hamiltonian reductions, but need O_n, Sp_{2n} instead of GL_d.

E.g. $\overline{\mathrm{Min}}(\mathfrak{so}_{2n}) =$ Ham. red

"Higgs branch" "$\boxed{D_n} — (C_1)$" means: $(\mathbb{C}^{2n} \otimes \mathbb{C}^2) /\!/\!/ Sp_2$.

{Recall: McKay $\Rightarrow$

$$\begin{tikzcd}[arrows=-] 1 \arrow[dr] & & & & & 1 \\ & 2 \arrow[r] & \cdots \arrow[r] & 2 \arrow[urr] \arrow[drr] & & \\ 1 \arrow[ur] & & & & & 1 \end{tikzcd}$$

gives $\underline{\text{Slice}}$ to codim 2 nilp orbit in $D_n = so(2n)$.

$GL_1 \times GL_1 \times GL_2 \times \ldots$

$\cong \mathbb{C}^2 / \text{binary dihedral}$.

$$\begin{tikzcd}[arrows=-] \boxed{n} \arrow[d] & & & \\ m_1 \arrow[r] & m_2 \arrow[r] & \cdots \arrow[r] & m_k \end{tikzcd}$$

$n > m_1 > \ldots > m_k \rightsquigarrow$

appropriate θ: $T^* G/P$, $G/P = \{\text{flags } \mathbb{C}^n \supseteq V_{m_1} \supseteq \ldots \supseteq V_{m_k}\}$

$\theta = 0$: Aff(thB), $\overline{\mathcal{O}}$. $\quad \dim V_i = i$.

<u>S3 varieties:</u> Given nilp orbit (closures)

$\overline{\mathcal{O}_\mu} \supseteq \mathcal{O}_\nu \rightsquigarrow S_{\mu\nu} =$ slice to $\mathcal{O}_\nu$ in $\overline{\mathcal{O}_\mu}$.

Also given by a type A quiver,

Hanany + Rudolph K. : give a description of quiver by <u>subtracting</u> gauge($\mathcal{O}_\mu$) − gauge($\mathcal{O}_\nu$),

[quiver was known] add appropriate framing.

Resolution of $S_{\mu\nu}$: take $\overline{\mathcal{O}_\mu} \supseteq \mathcal{O}_\nu$ "S3"

$$\begin{array}{ccc} \rho_\mu : T^*G/P & \twoheadrightarrow & \overline{\mathcal{O}_\mu} \\ & & \cup\!\!| \\ \rho_\mu^{-1}(S_{\mu\nu}) & \twoheadrightarrow & S_{\mu\nu} \end{array}$$

Symplectic $\Rightarrow$ crepant resolution.

⚠ in general types, not all $\overline{\mathcal{O}}$ admit s. resolutions, even need not be normal.

dim $S_{\mu\nu} = 2 \Rightarrow$ du Val sing. (A_n) [dim 2 s.sing $\Rightarrow$ du Val ADE].

Symplectic duality:

(special) symplectic leaf closures $\longleftrightarrow$ transverse slice to "complementary" leaf.

3D mirror symm

$$X \longleftrightarrow X^!$$

$$\cup \qquad\qquad\qquad \cup$$

$$\overline{\mathcal{O}} \longleftrightarrow \text{slice to } \mathcal{O}' \subseteq X^!$$

leaf closure

$$S_{\mu\nu} \longleftrightarrow S_{\nu^t\mu^t} \quad (X = X^! = Nil(\mathfrak{sl}_n))$$

Hyperplane arrangements

Thm (Namikawa) X conical symp sing, $X = \mathrm{Spec}(\mathcal{O}(X)) \leftarrow$ nonneg graded, $\mathbb{C} = \mathcal{O}(X)_0$.

$$HP^2(X) \cong H^2(\tilde{X}^{\circ})/W$$ parameterises all filtered Poisson deformations of $\mathcal{O}(X)$.

Here $\tilde{X} \twoheadrightarrow X$ is S.R. (or "(relative) minimal model" $\mathbb{Q}$-factorial terminalization).

$\tilde{X}^{\circ} \subseteq \tilde{X}$ smooth locus

Let's assume $\tilde{X}$ smooth.

Kaledin: Explained how construct univ. def. of

$\tilde{X}$: Take L = line bdle on $\tilde{X}$, (ample)

$c_1(L) \in H^2(\tilde{X})$

$\rightsquigarrow$ construct "twistor deformation"

$\tilde{\mathcal{X}}$, $\mathcal{L}$ deforming $\tilde{X}, L$ (one-parameter)

Symplectic form on total space of $\mathcal{L}$. "twistor space".

"Twistor": if $c_1(L)$ = kähler form of HK metric: base extends to $\mathbb{CP}^1$.

L ample $\Rightarrow$ $\mathcal{X}_t$ affine, ~~$t \neq 0$~~.

$W =$ "Namikawa Weyl group":

$H^2(\tilde{X}) = \oplus$ of refl. reps of W_i

$W =$ product of Weyl groups (finite)

$W = \prod_i W_i$

Has real form. $H^2(\tilde{X})_{\mathbb{R}}$.

<u>Note:</u> By Chevalley, refl rep$/W_i$ smooth, $\cong$ vector space

<u>E.g.</u> $\mathbb{C}^n / S_n \cong \mathbb{C}^n$ $(x-a_1)\cdots(x-a_n) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$.

$$
\begin{tikzcd}
\widetilde{\mathcal{H}} \arrow[r, two heads] \arrow[d] & \mathcal{H} \arrow[d] \\
H^2(\widetilde{X}) \arrow[r, two heads] & H^2(\widetilde{X})/W
\end{tikzcd}
$$

univ defos

Eg: $\tilde{\mathfrak{g}} = \{(\mathfrak{b}, x) \mid x \in \mathfrak{b}\}$

$$
\begin{tikzcd}
G \times_B \mathfrak{b} \arrow[r, two heads] \arrow[d] & \mathfrak{g} \arrow[d, "\text{Adjt quotient}"] \\
\mathfrak{h} \arrow[r, two heads] & \mathfrak{h}/W
\end{tikzcd}
$$

$\tilde{\mathfrak{g}} = G \times_B \mathfrak{b}$

$H^2(T^*G/B) = H^2(G/B) = \mathfrak{h}$

$\mathfrak{h}/W \cong H_P^2(\mathrm{Nil}(\mathfrak{g}))$

$$\mathcal{D} = \{ b \in H^2(\tilde{X}) \mid X_{w,b} \text{ is singular.} \}$$

analogue of $\mathfrak{h}$

$\cap$

$H^2(\tilde{X})$, $\mathcal{D}$ invt under W.

<u>Thm (Namikawa)</u>: $\mathcal{D}$ is a union of hyperplanes (can be called "Kähler root hyperplanes")

- $\exists$ fin. many crepant resols $/\cong$ of X (or minimal models)

$$H^2(\tilde{X})_{\mathbb{R}} \setminus \mathcal{D}_{\mathbb{R}} = \left\{ w(\mathrm{Amp}(\hat{X}_i)) \mid w \in W \right\}$$

real

$\hat{X}_i \twoheadrightarrow X$ resol.

$\Rightarrow$ can count # of crepant resols if can compute arrangement = # chambers / $|W|$

$\rightsquigarrow$ · Bellamy: $\mathbb{C}^{2n}/G$, $G < Sp_{2n}$ finite

· Bellamy–Craw: $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ $\Gamma \leq SL_2$ finite.

(all given as quiver vars, some Θ.

quiver var.

Cor: For Nil($\mathfrak{g}$), or any slice($\mathcal{O}_\lambda$ in Nil $\mathfrak{g}$),

Get Weyl arrangement: $\exists!$ crepant resol.

For du Val $\exists!$ min. resol.

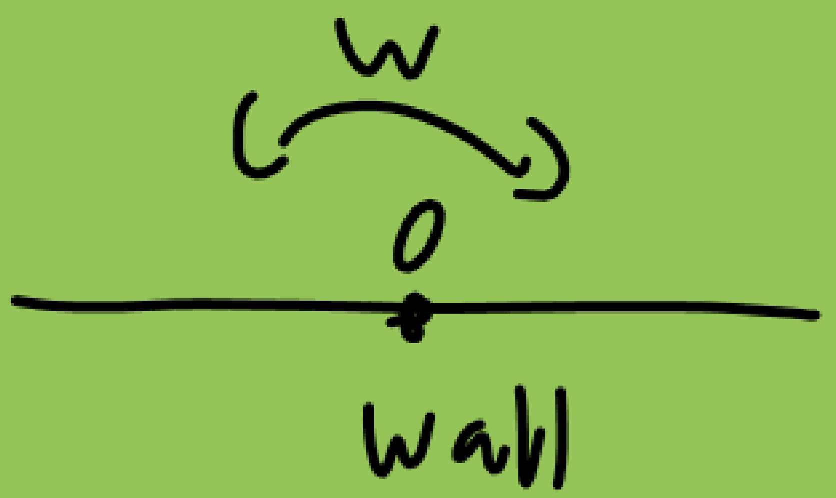
Crossing fixed plane for reflection $w \in W$

locally generically along plane: du Val sing.

$W = \prod_{i=1}^{m} W_i$, W_i = Weyl gp associated to codim-2 sing in X.

(or folding)

$$\begin{tikzcd}
T^*\mathbb{P}^1 \arrow[rr, phantom, "\simeq"] \arrow[dr] & & T^*\mathbb{P}^1 \arrow[dl] \\
& \mathrm{Nil}_{2\times 2} &
\end{tikzcd}$$



$$\begin{tikzcd}
\underset{V}{T^*Gr(n,m)} \arrow[dr] & & \underset{V^\perp}{T^*Gr(n,n-m)} \arrow[dl] \\
& \{A^2=0,\ \mathrm{rk}\,A \leq \min(m,n-m)\} \subseteq \mathrm{Mat}_n &
\end{tikzcd}$$

Resols $\cong$ (as resols) $\iff n=2m$.

Quiver vars: Thm (Bellamy–S): All symp sings, char. vars of closed surfaces classify when $\exists$ S.R.

[w/ A. Tirelli: sequel on multiplicative q. vars $\supseteq$ char vars of open sfc.

Also computed Weyl groups.

In progress w/ Craw: hope to classify all S.R.'s

$\rightsquigarrow$ Thm (McGerty–Nevins): Kirwan surjectivity:

$$H^*_G(pt) \cong H^*_G(\mu^{-1}(0)) \twoheadrightarrow H^*_G(\mu^{-1}(0)^{ss}) \cong H^*(\mathfrak{M}_{0,\theta}).$$

$\Rightarrow H^2(\mathfrak{M}_{0,\theta}) \cong \mathrm{Hom}(G, \mathbb{C}^*) \cong \mathbb{Z}^{Q_0}$

if smooth

Should say: All defs, all SR's come from quiver varieties.

(Rmk: $T^*\mathbb{P}^n \to \min(\mathfrak{sl}_{n+1})$

terminal, sing codim $= 2n$

$n \geq 2$

Not $\mathbb{Q}$-factorial.)

Symplectic Duality:

$$X \longleftrightarrow X^! \quad \text{"symplectic dual"}$$

(BLPW): $\underset{W \curvearrowright}{\mathcal{D}} \subseteq H^2(\tilde{X}) \longleftrightarrow \mathcal{E} \subseteq \mathrm{Hom}(\mathbb{C}^\times, \mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$

$\mathbb{T}$ = max $\underset{\sim}{\text{torus}}$ acting Ham. on $X^!$, commutes w/ dilations.

Quantisations: "Cat $\mathcal{O}$" Koszul dual

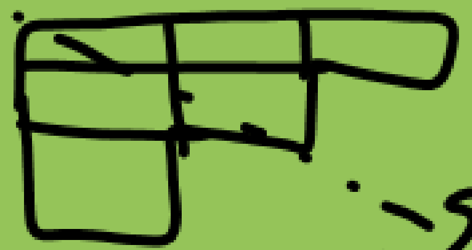
$\mathcal{O}_X \longleftrightarrow \mathcal{O}_{X^!}$.

infinite fixed locus

$W = N_G(\mathbb{T})/\mathbb{T}$.

Exs: $S_{\mu,\nu} \longleftrightarrow S_{\nu^t,\mu^t}$

μ,ν partitions of n, dual ν^t, μ^t



reflect to get dual.

$\cap|$

quiver var

$\cap|$

Higgs branch $\longleftrightarrow$ "Coulomb branch"

(Ham red $T^*V /\!/\!/ G$)

Hypertoric $\qquad$ [BLPW] $\qquad$ Gale dual

$$T^*\mathbb{C}^n /\!/\!/ (\mathbb{C}^*)^k \longleftrightarrow T^*\mathbb{C}^n /\!/\!/ (\mathbb{C}^*)^{k^+}$$

dim $2(n-k)$ $\qquad$ dim $2k$.

<u>Ex:</u>

$$T^*\mathbb{C}^n /\!/\!/ \mathbb{C}^\times \longleftrightarrow T^*\mathbb{C}^n /\!/\!/ (\mathbb{C}^*)^{n-1}$$

$T^*\mathbb{P}^{n-1}$, $\min(A_n)$ $\qquad$ type A_n du Val $\mathbb{C}^2/\mathbb{Z}/_{n+1}$

When you're lucky, a gauge theory has "mirror dual".

$$\mathrm{Higgs}(Q) \cong \mathrm{Coulomb}(Q^!)$$

[Q can even equal $Q^!$, e.g. 

]

$$\Rightarrow \mathrm{Higgs}(Q)^! \cong \mathrm{Higgs}(Q^!)$$

Happens for type A_n quiver vars $\supseteq S_{\mu\lambda}$.

Quantisation: X = conical symp sing

Thm (Losev): $H^2(\tilde{X})/W$ also parameterises filtered quantisations.

[e.g. $\mathcal{D}_\lambda(G/B) \twoheadrightarrow U\mathfrak{g}/\ker \chi_\lambda$)

$\Longrightarrow$ All quantisations of X are $\cong \Gamma(\mathcal{D})$, $\mathcal{D}$ quant. of $\tilde{X}$.

⚠ Γ Not always equiv but is for dominant enough.

What is Bernstein-Gelfand-Gelfand cat $\mathcal{O}$?

$\mathfrak{g}$ = f.d. ss Lie alg: allow ∞-dim rep,
but keep most nice properties

<u>Defn</u> Cat $\mathcal{O}$ = { reps where $\cdot\ \mathfrak{b} \subseteq \mathfrak{g}$ acts
locally finitely,
$\cdot\ \mathfrak{h} \subseteq \mathfrak{g}$ acts semisimply }

Fix $\chi : Z(U\mathfrak{g}) \to \mathbb{C}$, $\mathcal{O}_\chi = \{ V \mid \forall v \in V, (\ker \chi)^N \cdot v = 0, N \gg 0 \}$.

eg χ_0 = char of triv rep ($\ker = Z(U_{\mathfrak{g}}) \cap \mathfrak{g} \cdot U_{\mathfrak{g}}$)

$\mathcal{O}_{\chi_0}$ = "principal block"

$\mathfrak{g} = \mathfrak{sl}_2$

<u>Simples</u>: $\cdot\ \mathbb{C}$ $\cdot\ V_{-2} = \mathrm{Ind}_{\mathfrak{b} = \mathrm{span}(e,h)}^{\mathfrak{g}} \mathbb{C}_{-2}$

$= \mathrm{Span}(v, fv, f^2 v, \ldots)$

$h \cdot v = -2v$

<u>Thm</u> (Beilinson, Ginzburg, Soergel): $\mathcal{O}_0$ is Koszul self-dual.

Means: if we take simples L_i (fin many)

Ex: $\mathbb{C}$, V_{-2}

$$\mathcal{O} \simeq \operatorname{Ext}(\oplus L_i, \oplus L_i)\text{-mod}.$$

General: $\mathcal{O}, \mathcal{O}^!$ koszul dual if

$\mathcal{O}^! \simeq \operatorname{Ext}(\oplus L_i, \oplus L_i)$, vice-versa

$$\Rightarrow \mathcal{O} \overset{\text{der. eq.}}{\simeq} \mathcal{O}^!\text{-mod}.$$

[BLPW]: Define in general two versions of $\mathcal{O}$:

$\mathcal{O}_a$ "additive": take quant. D of X, conical s-s.

D-modules where D_+ acts loc. fin.

($\forall v$, $D_+ v$ f.d.)

$\mathcal{O}_g$ "geometric": $\mathcal{D}$-mods ($\Gamma(\mathcal{D}) = D$)

Support (set th) on attracting locus of $\mathbb{C}^\times$.

+ tech cond ($\exists$ lattice for $\mathcal{D}(0) \subseteq \mathcal{D}$)..

Statement: · $\mathcal{O}_g(\chi)^! \simeq \mathcal{O}_g(\chi^!)$

· If $\Gamma: \mathcal{O}_g \xrightarrow{\sim} \mathcal{O}_a$ equiv,
same for $\mathcal{O}_a$
(can ask Γ: D-mod $\to$ D-mod be equiv.)

Note: In g example, once we fixed χ,
fin. many simples in $\mathcal{O}$ $\Rightarrow$ fin. many f.d. irreps in D-mod.

In fact $\exists$ at most one:

every f.d. irrep of $\mathfrak{g} \xleftrightarrow{1-1}$ dominant highest weight

$\updownarrow$ 1-1

certain chars of $Z(U\mathfrak{g})$

[Etingof-S.]

<u>Thm</u>: More generally, if X has finitely many symplectic leaves, # fd irreps (D)

$\leq \dim HP_0(X) \to$ independent of D,

$\mathcal{O}(X)/\{\mathcal{O}(X), \mathcal{O}(X)\}$ <u>f.d.</u>